

T-STRUCTURES VIA RECOLLEMENTS FOR PIECEWISE HEREDITARY ALGEBRAS

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ABSTRACT. Following the work of Beilinson, Bernstein and Deligne, we study restriction and induction of t-structures in triangulated categories with respect to recollements. For derived categories of piecewise hereditary algebras we give a necessary and sufficient condition for a bounded t-structure to be induced from a recollement by derived categories of algebras. As a corollary we prove that for hereditary algebras of finite representation type all bounded t-structures can be obtained in this way.

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1. INTRODUCTION

The concepts of recollement and t-structure go back to the work of Beilinson, Bernstein and Deligne ([4]). In that paper, the interplay between these two ideas is explored in order to build the category of perverse sheaves. In fact, this category is seen as the heart of some t-structure induced from a recollement of the derived category of constructible sheaves on a stratified scheme. This technique of building a t-structure out of a recollement is the major topic of their paper.

One of the main results of [4] states that the heart of a t-structure is an abelian category. Of course every abelian category can be seen as the heart of a t-structure in its derived category (the standard t-structure) but the question of which abelian categories can be found as hearts of t-structures in a fixed derived category remains as an interesting research topic. In 1996, Happel, Reiten and Smalø ([12]) studied the interactions between t-structures and tilting theory, turning this concept into an important notion in the representation theory of finite dimensional algebras. Recently, interesting interactions between bounded t-structures for finite dimensional algebras and certain sets of *simple-minded* objects has appeared in the work of Keller and Nicolás ([14]) and Koenig and Yang ([15]). In their work, sets of simple-minded objects are used to construct equivalences of derived categories sending these sets to sets of simple objects.

The full set of bounded t-structures on a triangulated category is known only for a very few examples. It is known for division rings and semisimple algebras, although this seems to not yet be written. Thus, for the convenience of the reader, we include them in section 4. A famous (nontrivial) example for which much is known about t-structures is that of the Kroenecker quiver. In [8] one can find a complete list of bounded t-structures, up to autoequivalence, on the derived category of coherent sheaves over the projective line, which is well known to be

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equivalent to the derived category of finitely generated modules over the Kronecker quiver. In our last section (Example 6.6) we observe that the t-structure with heart $\text{coh}(\mathbb{P}^1)$ in this triangulated category cannot be induced from a recollement of derived categories. Moreover, the classification of bounded t-structures obtained in [8] for the Kronecker quiver is a consequence of the study of the space of stability conditions on the projective line. These ideas with origin in the work of Bridgeland ([5]) are of significant geometric interest and have been intensively studied for the past years.

On the other hand, recollements for derived categories and connections to tilting theory had been explored in the work of Parshall and Scott ([7]). More recently, Angeleri Hügel, Koenig and the first named author ([1]) have approached tilting theory using the general theory of recollements for triangulated categories. In that paper the authors provide ways of constructing tilting objects from recollements and constructing recollements from tilting modules. Our approach to this paper is of a similar spirit but with t-structures in mind instead of tilting objects. Also, in [2], Angeleri Hügel, Koenig and the first named author proceed to prove a Jordan Hölder theorem for derived categories of hereditary artin algebras, later generalised to piecewise hereditary algebras ([3]).

In this paper we study the interaction between these two key concepts: recollements and t-structures. We use the construction set by Beilinson, Bernstein and Deligne ([4]) to study properties of t-structures in the derived categories of piecewise hereditary algebras, using for that purpose some of the examples and techniques from [2, 3], namely how to construct a recollement from an indecomposable and exceptional (i.e. without self-extensions) object. Piecewise hereditary algebras were defined in [9] and [11]. An algebra is piecewise hereditary if it is derived equivalent to a certain hereditary abelian category. This covers, for example, hereditary algebras, (quasi-)tilted algebras and canonical algebras. Our main result states that for a piecewise hereditary algebra, the bounded t-structures whose heart is a length category can be obtained by a BBD-induction with respect to some recollement.

The paper is structured as follows. In section 2 we recall some preliminaries on recollements and t-structures that will be needed later. In section 3 we recall the definition of BBD-induction and restriction of t-structures with respect to a recollement and investigate general properties of an induced t-structure, namely nondegeneracy, boundedness (see also [6, 19]) and when it possesses a heart of finite length. Also, it is observed that, when restriction is possible, it is an inverse process to induction. In section 4 we discuss t-structures for semisimple algebras and in section 5 we prove that all bounded t-structures for A_n are induced with respect to a recollement associated with an idempotent. In the case $n = 2$ we can actually describe explicitly all the possible t-structures. Finally, section 6 proves the main result, using the techniques introduced in [14, 15].

2. PRELIMINARIES

In this section we recall the definition and some properties of recollements, (bounded) t-structures and simple-minded objects in triangulated categories.

2.1. Recollements. Let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be triangulated categories. \mathcal{D} is said to be a *recollement* of \mathcal{X} and \mathcal{Y} if there are six triangle functors as in the following diagram

$$\begin{array}{ccc} \xleftarrow{i^*} & \xleftarrow{j_!} & \\ \mathcal{Y} \xrightarrow{i_* = i_!} \mathcal{D} \xrightarrow{j^* = j^!} \mathcal{X} & & \\ \xleftarrow{i^!} & \xleftarrow{j_*} & \end{array}$$

such that

- (1) (i^*, i_*) , $(i_!, i^!)$, $(j_!, j^!)$, (j^*, j_*) are adjoint pairs;
- (2) i_* , j_* , $j_!$ are full embeddings;
- (3) $i^! \circ j_* = 0$ (and thus also $j^! \circ i_! = 0$ and $i^* \circ j_! = 0$);
- (4) for each $Z \in \mathcal{D}$ there are triangles

$$i_! i^! Z \rightarrow Z \rightarrow j_* j^* Z \rightarrow i_! i^! Z[1]$$

$$j_! j^! Z \rightarrow Z \rightarrow i_* i^* Z \rightarrow j_! j^! Z[1].$$

The following lemma follows easily from the definition of recollement and will be needed later.

Lemma 2.1. *Let \mathcal{D} be a triangulated category and Φ be an autoequivalence of \mathcal{D} , with Ψ its quasi-inverse. If*

$$\begin{array}{ccc} \xleftarrow{i^*} & \xleftarrow{j_!} & \\ \mathcal{Y} \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{X} & & \\ \xleftarrow{i^!} & \xleftarrow{j_*} & \end{array}$$

is a recollement, then so is

$$\begin{array}{ccc} \xleftarrow{i^* \Psi} & \xleftarrow{\Phi j_!} & \\ \mathcal{Y} \xrightarrow{\Phi i_*} \mathcal{D} \xrightarrow{j^* \Psi} \mathcal{X} & & \\ \xleftarrow{i^! \Psi} & \xleftarrow{\Phi j_*} & \end{array}$$

Proof. (Ψ, Φ, Ψ) is an adjoint triple and hence axiom (1) of recollement follows. Also, since Φ and Ψ are fully faithful, axiom (2) is automatically true as well. Now, by definition of quasi-inverse, $\Psi\Phi$ and $\Phi\Psi$ are naturally equivalent to the identity, implying that $i^! \Psi \Phi j_*$ is naturally equivalent to $i^! j_* = 0$, thus axiom (3). The same argument applies to check axiom (4), hence finishing the proof. \square

2.2. t-structures. For reference, see for example [4, 17]. A *t-structure* in a triangulated category \mathcal{D} is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories with the following properties: write $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ for $n \in \mathbb{Z}$,

- (1) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$, $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$;
- (2) $\text{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$;
- (3) for each Z in \mathcal{D} there is a distinguished triangle $X \rightarrow Z \rightarrow Y \rightarrow X[1]$ where $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

By definition, $\mathcal{D}^{\leq 0} = \{X \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(X, Y) = 0, \forall Y \in \mathcal{D}^{\geq 1}\}$ and $\mathcal{D}^{\geq 1} = \{Y \in \mathcal{D} : \text{Hom}_{\mathcal{D}}(X, Y) = 0, \forall X \in \mathcal{D}^{\leq 0}\}$. The subcategory $\mathcal{D}^{\leq 0}$ is called an *aisle*, and $\mathcal{D}^{\geq 0}$ is called a *coaisle*. The *heart* $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is always an abelian category. This is a full subcategory of \mathcal{D} . But in general its derived category has nothing to do with the original triangulated category \mathcal{D} . Nevertheless the first extension group is preserved.

Lemma 2.2 ([17] Chapter 4, 2.1.1). *Let \mathcal{D} be a triangulated category with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, and $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ be the heart. For any objects X, Y in \mathcal{A} it holds that $\text{Ext}_{\mathcal{A}}^1(X, Y) = \text{Hom}_{\mathcal{D}}(X, Y[1])$.*

For a given object $Z \in \mathcal{D}$, the canonical triangle as in (3) is unique (up to equivalence). The associated objects X and Y define the *truncation functors* $\tau_{\leq 0} : \mathcal{D} \rightarrow \mathcal{D}^{\leq 0}$ and $\tau_{\geq 1} : \mathcal{D} \rightarrow \mathcal{D}^{\geq 1}$. For $n \in \mathbb{Z}$, define $\tau_{\leq n} : \mathcal{D} \rightarrow \mathcal{D}^{\leq n}$ to be $[-n]\tau_{\leq 0}[n]$ and $\tau_{\geq n} : \mathcal{D} \rightarrow \mathcal{D}^{\geq n}$ to be $[-n+1]\tau_{\geq 1}[n-1]$. The composition $\tau_{\geq n}\tau_{\leq n}$ provides the *cohomological functor* H^n from \mathcal{D} to $\mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq n} = (\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0})[-n]$ and it can be shown to be equal to $H^n = [-n]H^0[n]$.

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and $(\tilde{\mathcal{D}}^{\leq 0}, \tilde{\mathcal{D}}^{\geq 0})$ be, respectively, t-structures in \mathcal{D} and $\tilde{\mathcal{D}}$, triangulated categories. An exact functor $F : \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ is said to be *right* (respectively *left*) *t-exact* with respect to these t-structures if $F(\mathcal{D}^{\leq 0}) \subset \tilde{\mathcal{D}}^{\leq 0}$ (respectively, if $F(\mathcal{D}^{\geq 0}) \subset \tilde{\mathcal{D}}^{\geq 0}$). The functor F is said to be *t-exact* if it is both left and right t-exact.

2.3. Nondegeneracy and boundedness. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in a triangulated category \mathcal{D} is said to be *nondegenerate* if

$$\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = 0 \quad \text{and} \quad \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\geq n} = 0$$

and it is *bounded* if

$$\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \mathcal{D} \quad \text{and} \quad \bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\geq n} = \mathcal{D}.$$

By [17, Chapter 4, 1.3.1 and 1.3.4], a t-structure is nondegenerate if and only if every object in \mathcal{D} with zero cohomology in every degree is isomorphic to the zero object, and it is bounded if and only if it is nondegenerate and every object in \mathcal{D} has bounded cohomology.

If a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is nondegenerate and the functors H^n , with $n \in \mathbb{Z}$, are the cohomological functors associated with it, it is easy to see that (check [17] for details):

$$\mathcal{D}^{\leq n} = \{X \in \mathcal{D} : H^i(X) = 0, \forall i > n\}$$

$$\mathcal{D}^{\geq n} = \{X \in \mathcal{D} : H^i(X) = 0, \forall i < n\}.$$

In the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category \mathcal{A} the standard t-structure is nondegenerate (and bounded if considered in $\mathcal{D}^b(\mathcal{A})$) and we denote it by $(\mathcal{D}_0^{\leq 0}, \mathcal{D}_0^{\geq 0})$.

For a class of objects $\mathcal{S} \subset \mathcal{D}$ we denote by $\text{tria } \mathcal{S}$ the smallest strictly full triangulated subcategory of \mathcal{D} containing \mathcal{S} .

Lemma 2.3. *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure in a triangulated category \mathcal{D} , and $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ be the heart. If the t-structure is bounded, then $\text{tria } \mathcal{A} = \mathcal{D}$. In particular, the Grothendieck group of \mathcal{A} coincides with that of \mathcal{D} .*

Proof. Suppose $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a bounded t-structure in \mathcal{D} . So every object X in \mathcal{D} has bounded cohomology (with respect to the t-structure). Using the truncation functors and the associated canonical triangles, we see that X is generated by its cohomology in finitely many steps. The statement follows. \square

2.4. Simple-minded objects. Let A be a finite dimensional \mathbb{K} -algebra. The bounded derived category of finite dimensional right A -modules will throughout be denoted by $\mathcal{D}^b(A)$.

Definition 2.4. A set X_1, \dots, X_n of objects in $\mathcal{D}^b(A)$ is a *family of simple-minded objects* if the following conditions hold:

- (1) $\text{Hom}(X_i, X_j[m]) = 0$, for all $m < 0$;
- (2) $\text{Hom}(X_i, X_j) = \delta_{ij}\mathbb{K}$, where δ_{ij} is the Kronecker delta;
- (3) The set generates $\mathcal{D}^b(A)$, i.e. $\text{tria}(X_1 \oplus \dots \oplus X_n) = \mathcal{D}^b(A)$.

Two families of simple-minded objects are called *equivalent* if they have the same closure under extensions.

Simple-minded objects play an important role when studying bounded t-structures of the derived category. Indeed the following holds.

Theorem 2.5 ([15] Corollary 3.9; [14] Corollary 11.5). *There is a bijection between the set of bounded t-structures in $\mathcal{D}^b(A)$ whose heart is a length category (i.e., every object has finite length) and the set of equivalence classes of families of simple-minded objects of $\mathcal{D}^b(A)$. In particular, the heart of a bounded t-structure in $\mathcal{D}^b(A)$ is a length category if and only if it is equivalent to a module category of some finite dimensional algebra.*

Given a bounded t-structure in $\mathcal{D}^b(A)$, if the heart is a length category, it is then equivalent to $\text{mod}(\Gamma)$ for some finite dimensional algebra Γ (which is constructed as the zero-th cohomology of the dg algebra $\tilde{\Gamma}$ in [15]). The correspondence in the theorem can be made explicit by associating to the bounded t-structure the set of simple Γ -modules. By Lemma 2.3, the rank of the Grothendieck group of the heart, which is the number of the simple Γ -modules, equals the rank of the Grothendieck group of $\mathcal{D}^b(A)$, and also of the algebra A . Conversely, given a family of simple-minded objects X_1, \dots, X_n , define the aisle (coaisle) to be the extension closure of all non-negative (non-positive) shifts of X_i 's. The extension closure of the X_i 's is just the heart of the associated t-structure. In particular the number n equals the rank of the Grothendieck group of A .

We also have the following useful lemma.

Lemma 2.6. *Suppose \mathcal{A} is a length abelian category with finitely many simple objects X_1, \dots, X_n . Let $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure in $\mathcal{D}^b(\mathcal{A})$. Then $\mathcal{D}^{\leq 0}$ contains a shift of the standard aisle.*

Proof. Since the t-structure is bounded, for each $1 \leq i \leq n$, there is an integer k_i such that $X_i \in \mathcal{D}^{\leq k_i}$. Define

$$k := \min \{k_i : 1 \leq i \leq n\}$$

and then it is clear that $X_1, \dots, X_n \in \mathcal{D}^{\leq k}$. Since \mathcal{A} is the extension closure of X_1, \dots, X_n and an aisle is closed under extensions, we have that $\mathcal{A} \subset \mathcal{D}^{\leq k}$. By definition of t-structure, if $X \in \mathcal{D}^{\leq n}$ for some $n \in \mathbb{Z}$ then $X[i] \in \mathcal{D}^{\leq n}$ for all $i \geq 0$, and thus $\mathcal{D}_0^{\leq 0} \subset \mathcal{D}^{\leq k}$, or equivalently, $\mathcal{D}_0^{\leq -k} \subset \mathcal{D}^{\leq 0}$. \square

3. BBD-INDUCTION AND BBD-RESTRICTION

Throughout, \mathcal{D} will be a triangulated category. We denote the set of all t-structures in \mathcal{D} by $\mathcal{T}_{\mathcal{D}}$. We shall assume, unless otherwise stated, that this triangulated category has a recollement, denoted by \mathcal{R} , by triangulated categories \mathcal{X} and

\mathcal{Y} as follows:

$$\begin{array}{ccc} & \xleftarrow{i^*} & \xleftarrow{j^!} \\ \mathcal{Y} & \xrightarrow{i_*} \mathcal{D} & \xrightarrow{j^*} \mathcal{X} \\ & \xleftarrow{i^!} & \xleftarrow{j_*} \end{array}$$

The theorem that motivates our approach is the following.

Theorem 3.1 ([4] Theorem 1.4.10). *Suppose $T_{\mathcal{X}} = (\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$ and $T_{\mathcal{Y}} = (\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ are t-structures in \mathcal{X} and \mathcal{Y} respectively. Then $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ defined by:*

$$\mathcal{D}^{\leq 0} := \{Z \in \mathcal{D} : j^*Z \in \mathcal{X}^{\leq 0}, i^*Z \in \mathcal{Y}^{\leq 0}\}$$

$$\mathcal{D}^{\geq 0} := \{Z \in \mathcal{D} : j^*Z \in \mathcal{X}^{\geq 0}, i^!Z \in \mathcal{Y}^{\geq 0}\}$$

is a t-structure in \mathcal{D} .

We will use the notation $T =: \text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$ and call T the *BBD-induction* (or simply *induction*) of $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$.

The problem of restricting t-structures is a more delicate issue and it is also discussed in [4]. We include the proof for sake of completion.

Proposition 3.2 ([4] Proposition 1.4.12). *Let $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure in \mathcal{D} . The following are equivalent:*

- (1) $j_!j^*$ is right t-exact, i.e., $j_!j^*\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}$;
- (2) j_*j^* is left t-exact, i.e., $j_*j^*\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq 0}$;
- (3) T is induced with respect to the recollement \mathcal{R} .

Proof. The fact that the first two statements are equivalent since $(j_!j^*, j_*j^*)$ is an adjoint pair. Note that these two conditions imply that i_*i^* is right t-exact and $i_*i^!$ is left t-exact by the canonical triangles of the recollement. This implies, in particular, that $\text{Hom}(i^*\mathcal{D}^{\leq 0}, i^!\mathcal{D}^{\geq 1}) = 0$.

It is clear that (3) implies (1) (and thus (2)). Indeed, if $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is induced with respect to the recollement, then by definition $j^*\mathcal{D}^{\leq 0} \subset \mathcal{X}^{\leq 0}$. Also, $j_!\mathcal{X}^{\leq 0} \subset \mathcal{D}^{\leq 0}$ since, by adjunction, for $X \in \mathcal{X}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ (which implies that $j^*Y \in \mathcal{X}^{\geq 1}$),

$$\text{Hom}_{\mathcal{D}}(j_!X, Y) = \text{Hom}_{\mathcal{X}}(X, j^*Y) = 0.$$

Conversely, suppose that (1) (and (2)) hold. Then it is easy to check that $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ is a t-structure in \mathcal{X} (and thus j^* is t-exact). Observe that $i_*^{-1}(i_*\mathcal{Y} \cap \mathcal{D}^{\leq 0}) = i^*\mathcal{D}^{\leq 0}$ and $i_*^{-1}(i_*\mathcal{Y} \cap \mathcal{D}^{\geq 0}) = i^!\mathcal{D}^{\geq 0}$, since i_*i^* is right t-exact and $i_*i^!$ is left t-exact. We check now that the pair $(i_*^{-1}(i_*\mathcal{Y} \cap \mathcal{D}^{\leq 0}), i_*^{-1}(i_*\mathcal{Y} \cap \mathcal{D}^{\geq 0}))$ is a t-structure in \mathcal{Y} . Since i_* is a full embedding, it is enough to check that $(i_*\mathcal{Y} \cap \mathcal{D}^{\leq 0}, i_*\mathcal{Y} \cap \mathcal{D}^{\geq 0})$ is a t-structure in $i_*\mathcal{Y}$. The orthogonality condition follows from the observations above. We only need to check the triangle condition of a t-structure. Let $Y \in \mathcal{Y}$. Since j^* is t-exact then $j^*\tau^{\leq 0}i_*Y = \tau_{\mathcal{X}}^{\leq 0}j^*i_*Y = 0$ and similarly, $j^*\tau^{\geq 1}i_*Y = 0$. Thus the truncations $\tau^{\leq 0}i_*Y$ and $\tau^{\geq 1}i_*Y$ lie in the image of i_* and thus we get a triangle

$$\tau^{\leq 0}i_*Y \rightarrow i_*Y \rightarrow \tau^{\geq 1}i_*Y \rightarrow \tau^{\leq 0}i_*X[1]$$

in which $\tau^{\leq 0}i_*Y \in i_*\mathcal{Y} \cap \mathcal{D}^{\leq 0}$ and $\tau^{\geq 1}i_*Y \in i_*\mathcal{Y} \cap \mathcal{D}^{\geq 1}$.

It remains to check that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is BBD-induced with respect to \mathcal{R} . We prove that indeed it is induced with respect to \mathcal{R} by taking $T_{\mathcal{X}} := (j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ in \mathcal{X} and $T_{\mathcal{Y}} := (i^*\mathcal{D}^{\leq 0}, i^!\mathcal{D}^{\geq 0})$ in \mathcal{Y} . Let $\tilde{T} = \text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$. Clearly \tilde{T} is determined by the aisle

$$\tilde{\mathcal{D}}^{\leq 0} := \{Z \in \mathcal{D} : j^*Z \in j^*\mathcal{D}^{\leq 0}, i^*Z \in i^*\mathcal{D}^{\leq 0}\}$$

and it is clear that $\mathcal{D}^{\leq 0} \subset \tilde{\mathcal{D}}^{\leq 0}$. We prove that $\tilde{\mathcal{D}}^{\leq 0} = \mathcal{D}^{\leq 0}$, thus proving that $\tilde{T} = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. If $Z \in \tilde{\mathcal{D}}^{\leq 0}$, $j_!j^*Z \in \mathcal{D}^{\leq 0}$ and $i_*i^*Z \in \mathcal{D}^{\leq 0}$ since $j_!$ and i_* are right t-exact with respect to $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ and $(i^*\mathcal{D}^{\leq 0}, i^*\mathcal{D}^{\geq 0})$. Thus, the triangle

$$j_!j^!Z \longrightarrow Z \longrightarrow i_*i^*Z \longrightarrow j_!j^!Z[1],$$

given by the recollement, proves that $Z \in \mathcal{D}^{\leq 0}$. \square

Definition 3.3. Fix a recollement \mathcal{R} for \mathcal{D} as above. A t-structure $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in \mathcal{D} is said to be *compatible with the recollement* if $j_!j^*$ is right t-exact. We denote by $\mathcal{T}_{\mathcal{D}}^{\mathcal{R}}$ the set of t-structure in \mathcal{D} which are compatible with the recollement \mathcal{R} . For $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \in \mathcal{T}_{\mathcal{D}}^{\mathcal{R}}$ we define the *BBD-restriction* (or simply *restriction*) of T with respect to \mathcal{R} to be the pair $\text{Res}(T) := (T_{\mathcal{X}}, T_{\mathcal{Y}}) \in \mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}}$ where $T_{\mathcal{X}} := (j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ and $T_{\mathcal{Y}} := (i^*\mathcal{D}^{\leq 0}, i^*\mathcal{D}^{\geq 0})$.

It follows from the above proof that the aisle and the coisles of $T_{\mathcal{Y}}$ are respectively given by

$$\begin{aligned} i^*\mathcal{D}^{\leq 0} &= i_*^{-1}(i_*\mathcal{Y} \cap \mathcal{D}^{\leq 0}) = \{Y \in \mathcal{Y} : i_*Y \in \mathcal{D}^{\leq 0}\}, \\ i^!\mathcal{D}^{\geq 0} &= i_*^{-1}(i_*\mathcal{Y} \cap \mathcal{D}^{\geq 0}) = \{Y \in \mathcal{Y} : i_*Y \in \mathcal{D}^{\geq 0}\}. \end{aligned}$$

Also if $T_{\mathcal{Y}}$, T and $T_{\mathcal{X}}$ are compatible t-structures with respect to the recollement \mathcal{R} , then the functors i^* and $j_!$ are right t-exact, i_* and j^* are t-exact, and $i^!$ and j_* are left t-exact.

The proposition above provides us with the following two maps

$$\text{Res} : \mathcal{T}_{\mathcal{D}}^{\mathcal{R}} \longrightarrow \mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}}$$

$$\text{Ind} : \mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}} \longrightarrow \mathcal{T}_{\mathcal{D}}$$

and shows that for a t-structure $T \in \mathcal{T}_{\mathcal{D}}^{\mathcal{R}}$, $\text{Ind}(\text{Res}(T)) = T$. Indeed we have more.

Corollary 3.4. Let \mathcal{R} be a recollement as before and let $T_{\mathcal{X}} = (\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$ and $T_{\mathcal{Y}} = (\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ be t-structures in \mathcal{X} and \mathcal{Y} respectively. We have that $\text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$ is compatible with \mathcal{R} and $\text{Res}(\text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})) = (T_{\mathcal{X}}, T_{\mathcal{Y}})$. Consequently, Res and Ind are inverse bijections between $\mathcal{T}_{\mathcal{D}}^{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}}$.

Proof. Let $T := \text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}}) = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. It is compatible with \mathcal{R} by Proposition 3.2. Let $\text{Res}(T) = ((\tilde{\mathcal{X}}^{\leq 0}, \tilde{\mathcal{X}}^{\geq 0}), (\tilde{\mathcal{Y}}^{\leq 0}, \tilde{\mathcal{Y}}^{\geq 0}))$. Clearly,

$$\tilde{\mathcal{X}}^{\leq 0} := j^*(\mathcal{D}^{\leq 0}) \subset \mathcal{X}^{\leq 0} \quad \text{and} \quad \tilde{\mathcal{X}}^{\geq 1} := j^*\mathcal{D}^{\geq 1} \subset \mathcal{X}^{\geq 1},$$

by definition of induced t-structure, and thus $\mathcal{X}^{\leq 0} = \tilde{\mathcal{X}}^{\leq 0}$. On the other hand, let $Y \in \tilde{\mathcal{Y}}^{\leq 0} = \{Y \in \mathcal{Y} : i_*Y \in \mathcal{D}^{\leq 0}\}$. Then $Y \cong i^*i_*Y \in \mathcal{Y}^{\leq 0}$ by definition of $\mathcal{D}^{\leq 0}$. Also, if $Y \in \mathcal{Y}^{\leq 0}$, then so is i^*i_*Y (because it is isomorphic to Y). Therefore $i_*Y \in \mathcal{D}^{\leq 0}$, which means that $Y \in \tilde{\mathcal{Y}}^{\leq 0}$. \square

This means in particular that once we have a description of all t-structures of the middle term of a recollement, then we have a description of all t-structures of the left and the right hand sides of the recollement.

We now discuss some properties of t-structures that are preserved via induction and restriction. Nondegeneracy is discussed in [4] and boundedness is partly discussed in [6, 19].

Lemma 3.5. The bijections Res and Ind restrict to bijections between nondegenerate (respectively, bounded) t-structures in $\mathcal{T}_{\mathcal{D}}^{\mathcal{R}}$ and $\mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}}$.

Proof. Suppose $(T_{\mathcal{X}}, T_{\mathcal{Y}}) \in \mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}}$ is a pair of nondegenerate t-structures. Let us check that the nondegeneracy of its BBD-induction $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) := \text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$: If $Z \in \mathcal{D}^{\leq n}$ (respectively, $Z \in \mathcal{D}^{\geq n}$) for all $n \in \mathbb{Z}$, then $i^*Z \in \mathcal{Y}^{\leq n}$ (respectively, $i^!Z \in \mathcal{Y}^{\geq n}$) and $j^*Z \in \mathcal{X}^{\leq n}$ (respectively, $j^*Z \in \mathcal{X}^{\geq n}$) for all $n \in \mathbb{Z}$, implying that $i^*Z = j^*Z = 0$ (respectively, $i^!Z = j^!Z = 0$). By the canonical triangles induced by the recollement we get $Z = 0$ and hence the induced t-structure is nondegenerate. Moreover suppose $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$ are bounded. For $Z \in \mathcal{D}$, because of the boundedness of $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$, there exists integers m, n, k and l such that $j^*Z \in \mathcal{X}^{\leq m}$ and $\mathcal{X}^{\geq n}$, $i^*Z \in \mathcal{Y}^{\leq k}$ and $i^!Z \in \mathcal{Y}^{\geq l}$. It follows from the canonical triangles

$$\begin{aligned} j_!j^!Z &\longrightarrow Z \longrightarrow i_*i^*Z \longrightarrow j_!j^!Z[1] \\ i_!i^!Z &\longrightarrow Z \longrightarrow j_*j^*Z \longrightarrow i_!i^!Z[1] \end{aligned}$$

and from the (left or right) t-exactness of the functors i_* , j_* and $j_!$ that $Z \in \mathcal{D}^{\leq \max\{m, k\}}$ and $Z \in \mathcal{D}^{\geq \min\{n, l\}}$. Thus the induced t-structure is bounded as well.

Conversely, let $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a nondegenerate t-structure in \mathcal{D} compatible with \mathcal{R} . To show the nondegeneracy of the restricting t-structure, we need to show that $\cap_{n \in \mathbb{Z}} i^*\mathcal{D}^{\leq n} = 0 = \cup_{n \in \mathbb{Z}} i^!\mathcal{D}^{\geq n}$ and $\cup_{n \in \mathbb{Z}} j^*\mathcal{D}^{\leq n} = 0 = \cup_{n \in \mathbb{Z}} j^!\mathcal{D}^{\geq n}$. If $Y \in i^*\mathcal{D}^{\leq n}$ for all $n \in \mathbb{Z}$, then $i_*Y \in i_*i^*\mathcal{D}^{\leq n}$ for all $n \in \mathbb{Z}$. Since i_*i^* is right t-exact, $i_*Y \in \cap_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = 0$. Hence i_*Y and also Y is trivial. The other equalities follows by similar argument.

Moreover suppose now T is bounded. To show the boundedness of restricting t-structures, we need to show $\cup_{n \in \mathbb{Z}} i^*\mathcal{D}^{\leq n} = \mathcal{Y} = \cup_{n \in \mathbb{Z}} i^!\mathcal{D}^{\geq n}$ and $\cup_{n \in \mathbb{Z}} j^*\mathcal{D}^{\leq n} = \mathcal{X} = \cup_{n \in \mathbb{Z}} j^!\mathcal{D}^{\geq n}$. For $Y \in \mathcal{Y}$, because of the boundedness of T , there exists integers s and t such that $i_*Y \in \mathcal{D}^{\leq s}$ and $\mathcal{D}^{\geq t}$. Hence $Y \cong i^*i_*Y$ belongs to $i^*\mathcal{D}^{\leq s}$ and $Y \cong i^!i_*Y$ belongs to $i^!\mathcal{D}^{\geq t}$. Similarly for $X \in \mathcal{X}$, there exists integers k and l such that $j_!X \in \mathcal{D}^{\leq k}$ and $j_*X \in \mathcal{D}^{\geq l}$. It follows that $X \cong j^*j_!X \in j^*\mathcal{D}^{\leq k}$ and $X \cong j^*j_*X \in j^*\mathcal{D}^{\geq l}$. \square

There is a natural action of the autoequivalent group $\text{Aut}(\mathcal{D})$ on the set $\mathcal{T}_{\mathcal{D}}$ of all t-structures in \mathcal{D} . Two t-structures T_1 and T_2 are called *equivalent*, if they are transformed to each other by an autoequivalence of \mathcal{D} , in other words, the (co-)aisle of T_1 is sent to the (co-)aisle of T_2 by some $\Phi \in \text{Aut}(\mathcal{D})$.

Lemma 3.6. *Let T_1 and T_2 be two equivalent t-structures. Then T_1 is a BBD-induction if and only if so is T_2 .*

Proof. Suppose $\Phi \in \text{Aut}(\mathcal{D})$ sends T_1 to T_2 . Assume T_1 is a BBD-induction with respect to a recollement

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{Y} & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{X} \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

This means T_1 is compatible with the recollement. By Lemma 2.1, the autoequivalence Φ induces a new recollement

$$\begin{array}{ccccc} & \xleftarrow{i^*\Psi} & & \xleftarrow{\Phi j_!} & \\ \mathcal{Y} & \xrightarrow{\Phi i_*} & \mathcal{D} & \xrightarrow{j^*\Psi} & \mathcal{X} \\ & \xleftarrow{i^!\Psi} & & \xleftarrow{\Phi j_*} & \end{array}$$

where Ψ is the quasi-inverse of Φ . It is straightforward to check that T_2 is compatible with this new recollement. The lemma follows now from Corollary 3.4. \square

As mentioned in the preliminaries section, bounded t-structures whose heart is a length category with finitely many simple objects are of particular interest when dealing with derived categories of finite dimensional algebras. A useful relation between the hearts of t-structures in \mathcal{X} and \mathcal{Y} and the heart of the t-structure in \mathcal{D} is established in [4].

Proposition 3.7 ([4] Proposition 1.4.18). *Let $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ be the hearts of $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$, respectively, for a pair $(T_{\mathcal{X}}, T_{\mathcal{Y}}) \in \mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}}$. Also, let \mathcal{A} be the heart of $\text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$. Then $\mathcal{A}_{\mathcal{X}} \cong \mathcal{A}/i_*\mathcal{A}_{\mathcal{Y}}$ as abelian categories.*

Remark 3.8. Note that, in the notation of the proposition, $j^*(\mathcal{A}) \subset \mathcal{A}_{\mathcal{X}}$ and $i_*(\mathcal{A}_{\mathcal{Y}}) \subset \mathcal{A}$. This follows from the definitions of restriction and induction and Proposition 3.4. Moreover, if we identify \mathcal{Y} with its image by i_* , then we get $\mathcal{A}_{\mathcal{Y}} \cong \mathcal{A} \cap \text{Im}(i_*)$.

The following result will be used later.

Proposition 3.9. *Let $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ be the hearts of $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$, respectively, for a pair $(T_{\mathcal{X}}, T_{\mathcal{Y}}) \in \mathcal{T}_{\mathcal{X}} \times \mathcal{T}_{\mathcal{Y}}$. Also, let \mathcal{A} be the heart of $\text{Ind}(T_{\mathcal{X}}, T_{\mathcal{Y}})$. Then $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ are length categories if and only if so is \mathcal{A} .*

Proof. Suppose $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ are length hearts and let $Z \in \mathcal{A}$ be an object of infinite length, i.e., there exists an infinite sequence of strict monomorphisms of \mathcal{A}

$$\cdots \xrightarrow{f_{k-2}} Z_{k-1} \xrightarrow{f_{k-1}} Z_k \xrightarrow{f_k} Z_{k+1} \xrightarrow{f_{k+1}} \cdots \longrightarrow Z,$$

where k runs over an index set $K \subset \mathbb{Z}$ of consecutive integers, whose image by j^* is a chain of monomorphisms

$$\cdots \xrightarrow{j^* f_{k-2}} j^* Z_{k-1} \xrightarrow{j^* f_{k-1}} j^* Z_k \xrightarrow{j^* f_k} j^* Z_{k+1} \xrightarrow{j^* f_{k+1}} \cdots \longrightarrow j^* Z$$

that cannot be strict since $\mathcal{A}_{\mathcal{X}}$ is a length category. Since j^* is exact, this means that $j^* X_k \neq 0$ for only finitely many $k \in K$, where $X_k := \text{coker}(f_k)$. Let us say that $j^* X_k = 0$ for all $k \in K$ such that $k \leq a$ and $k \geq b$ for some $a, b \in K$. Hence, $K_{\leq a} := \{k \in K : k \leq a\}$ is infinite or $K_{\geq b} := \{k \in K : k \geq b\}$ is infinite. Suppose $K_{\leq a}$ is infinite and consider the objects $Y_l := \text{coker}(Z_{a-l} \rightarrow Z_a)$, for $l \in \mathbb{N}$, where the map is the composition $f_{a-l} f_{a-l+1} \cdots f_{a-1}$. Clearly $j^*(Y_l) = 0$ (since $j^*(Z_{a-l}) \cong j^*(Z_a)$) and hence $Y_l \in \mathcal{H} \cap \text{Im}(i_*)$. But this means we have a strict chain of epimorphisms

$$\cdots \rightarrow Y_{l+1} \rightarrow Y_l \rightarrow Y_{l-1} \rightarrow \cdots \rightarrow Y_1$$

which is a contradiction since $\mathcal{A}_{\mathcal{Y}} \cong \mathcal{A} \cap \text{Im}(i_*)$ is a length category. Similarly, if $K_{\geq b}$ is infinite, then one can define $Y_l := \text{coker}(Z_b \rightarrow Z_{b+l})$ and the same argument will hold.

Conversely, if \mathcal{A} is a length heart, then it is easy to check that so is every full subcategory and every quotient category of \mathcal{A} . Then, by Proposition 3.7 $\mathcal{A}_{\mathcal{Y}}$ and $\mathcal{A}_{\mathcal{X}}$ are length categories. \square

Remark 3.10. We can, moreover, say something about the simple objects of the induced heart, as stated in [4, Proposition 1.4.26]. In the notation of the above proposition, it is easy to see that the adjunction morphisms given by the recollement induce a natural transformation of functors from $\mathcal{A}_{\mathcal{X}}$ to \mathcal{A}

$$\phi : H^0 j_! \epsilon \rightarrow H^0 j_* \epsilon$$

where H^0 is the cohomological functor associated to the induced t-structure T and ϵ is the natural embedding of $\mathcal{A}_{\mathcal{X}}$ in \mathcal{X} . Define the functor $j_{!*}$ from $\mathcal{A}_{\mathcal{X}}$ to \mathcal{A} by setting $j_{!*}(X) := \text{Im } \phi(X)$. Then it can be proved that the simple objects of \mathcal{A} are of the form i_*Y for Y simple in \mathcal{Y} or of the form $j_{!*}X$ for X simple in \mathcal{X} .

4. T-STRUCTURES FOR SEMISIMPLE ALGEBRAS

In this section we classify t-structures for semisimple algebras.

Let \mathcal{D}_1 and \mathcal{D}_2 be two triangulated categories with t-structures $T_1 = (\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$ and $T_2 = (\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$ respectively. It is clear that the direct sum $T_1 \oplus T_2$, defined to be $(\mathcal{D}_1^{\leq 0} \oplus \mathcal{D}_2^{\leq 0}, \mathcal{D}_1^{\geq 0} \oplus \mathcal{D}_2^{\geq 0})$, is a t-structure in $\mathcal{D}_1 \oplus \mathcal{D}_2$. Conversely all t-structures in $\mathcal{D}_1 \oplus \mathcal{D}_2$ arise in this way.

Lemma 4.1. *Every t-structure in the direct sum $\mathcal{D}_1 \oplus \mathcal{D}_2$ is of the form*

$$(\mathcal{D}_1^{\leq 0} \oplus \mathcal{D}_2^{\leq 0}, \mathcal{D}_1^{\geq 0} \oplus \mathcal{D}_2^{\geq 0}),$$

where $(\mathcal{D}_1^{\leq 0}, \mathcal{D}_1^{\geq 0})$ and $(\mathcal{D}_2^{\leq 0}, \mathcal{D}_2^{\geq 0})$ are t-structures in \mathcal{D}_1 and \mathcal{D}_2 respectively.

Proof. Consider the recollement

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ & & \mathcal{D} & & \\ \mathcal{D}_1 & \xrightarrow{i_*} & & \xrightarrow{j^*} & \mathcal{D}_2 \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}$$

where $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$, $i_* : \mathcal{D}_1 \rightarrow \mathcal{D}$ and $j_! = j_* : \mathcal{D}_2 \rightarrow \mathcal{D}$ are canonical full embeddings, and $i^* = i^! : \mathcal{D} \rightarrow \mathcal{D}_1$ and $j^* : \mathcal{D} \rightarrow \mathcal{D}_2$ are canonical projections. The BBD-induction of t-structures T_1 in \mathcal{D}_1 and T_2 in \mathcal{D}_2 is exactly the direct sum $T_1 \oplus T_2$. Conversely, by Corollary 3.4, to show a t-structure $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in \mathcal{D} arises in this way, it suffices to show it is compatible with the given recollement, namely that $j_!j^*\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}$. Now, given an object $X \in \mathcal{D}^{\leq 0}$, it is a direct sum of $X_1 \oplus X_2$ for $X_i \in \mathcal{D}_i$ ($i = 1, 2$). The aisle $\mathcal{D}^{\leq 0}$ is the left perpendicular category of $\mathcal{D}^{\geq 1}$ (with respect to the pairing $\text{Hom}_{\mathcal{D}}(-, -)$) and thus $\mathcal{D}^{\leq 0}$ is closed under taking direct summand. Hence $j_!j^*(X) = j_!(X_2) = X_2$ belongs to $\mathcal{D}^{\leq 0}$, as wanted. \square

Let \mathbb{K} be any field. The Artin-Wedderburn's theorem says finite dimensional semisimple \mathbb{K} -algebras are built up by matrix algebras over division rings. In other words they are of the form $M_{n_1}(\mathbb{D}_1) \times \dots \times M_{n_s}(\mathbb{D}_s)$, where \mathbb{D}_i is a division ring over \mathbb{K} and n_i a natural number (for all i).

Lemma 4.2. *Let \mathbb{D} be a division ring. Then a t-structure in $\mathcal{D}^b(\mathbb{D})$ is either trivial (i.e., it has a trivial aisle or a trivial coaisle) or it is a shift of the standard t-structure.*

Proof. The only indecomposable objects of $\mathcal{D}^b(\mathbb{D})$ are $\mathbb{D}[p]$ for $p \in \mathbb{Z}$. If the t-structure is non-trivial, then there is a minimal p_0 such that $\mathbb{D}[p_0] \in \mathcal{D}^{\leq 0}$ (and thus $\mathbb{D}[p] \in \mathcal{D}^{\leq 0}$ for all $p \geq p_0$). Clearly no other indecomposable can lie in $\mathcal{D}^{\leq 0}$ by minimality of p_0 and thus $\mathcal{D}^{\leq 0}$ coincides with $\mathcal{D}_0^{\leq 0}[-p_0]$, where $\mathcal{D}_0^{\leq 0}$ is the standard aisle for $\mathcal{D}^b(\mathbb{D})$. \square

We identify $\mathcal{T}_{\mathcal{D}^b(\mathbb{D})}$ with the set $\bar{\mathbb{Z}} := \mathbb{Z} \cup \{-\infty, +\infty\}$ where an integer $n \in \mathbb{Z}$ represents the t-structure associated with the aisle $\mathcal{D}_0^{\leq n}$, $-\infty$ represents the trivial t-structure given by $\mathcal{D}^{\leq 0} = 0$ and $+\infty$ represents the trivial t-structure given by $\mathcal{D}^{\leq 0} = \mathcal{D}^b(\mathbb{D})$. Those t-structures indexed by $n \in \mathbb{Z}$ are bounded, and

they are equivalent by shifts in the derived category $\mathcal{D}^b(\mathbb{D})$. So, up to derived autoequivalences, there are only three t-structures in $\mathcal{T}_{\mathcal{D}^b(\mathbb{D})}$ and there is only one among those that is bounded.

Corollary 4.3. *Let $A = M_{n_1}(\mathbb{D}_1) \times \dots \times M_{n_s}(\mathbb{D}_s)$ be a semisimple algebra over \mathbb{K} with s blocks. Then the t-structures in the derived category $\mathcal{D}^b(A)$ of A is indexed by $\bar{\mathbb{Z}} \times \dots \times \bar{\mathbb{Z}}$ (s times). Among them those indexed by $\mathbb{Z} \times \dots \times \mathbb{Z}$ (s times) are bounded.*

Remark 4.4. A similar observation to the above allows us to conclude that, up to derived autoequivalences, there is only one bounded t-structure. Indeed, for any n-tuple of integers, there is an autoequivalence of the derived category taking the correspondent t-structure to the standard one - and this autoequivalence is just built from a suitable choice of triangulated shifts in each component.

5. BOUNDED T-STRUCTURES IN A_n

In this section we prove that any bounded t-structure in $\mathcal{D}^b(A_n)$, where A_n is the path algebra of

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \dots \longrightarrow \bullet_{n-1} \longrightarrow \bullet_n$$

over a field \mathbb{K} , is BBD-induced with respect to the recollement associated with an idempotent e_r ($r = 1, 2, \dots, n$). Recall that given an idempotent e on a hereditary algebra A we have a recollement \mathcal{R}_e (sometimes denoted \mathcal{R}_r if A is a path algebra and $e = e_r$) of the form ([7, 1]):

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j!} & \\ \mathcal{D}^b(A/AeA) & \xrightarrow{i_*} & \mathcal{D}^b(A) & \xrightarrow{j^*} & \mathcal{D}^b(eAe) \\ & \xleftarrow{i!} & & \xleftarrow{j_*} & \end{array}$$

where the functors can all be explicitly described as derived functors. For our purposes we just need

$$j^* = - \otimes_A^L Ae, \quad j! = - \otimes_{eAe}^L A.$$

Theorem 5.1. *For any bounded t-structure $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in $\mathcal{D}^b(A_n)$, there is an idempotent e_r ($1 \leq r \leq n$) such that T is compatible with \mathcal{R}_r .*

Proof. We start with a simple homological computation. Since the algebra is hereditary we represent any indecomposable object by its projective resolution. Note that in such a projective resolution each projective has atmost multiplicity one. Also, we use the convention $P_0 = 0$. For an idempotent r , denote the functors in \mathcal{R}_r by j_l^r, j_r^* . We observe that for $1 \leq k \leq n$, $0 \leq l \leq n-1$ we have

$$j_l^r j_r^*(P_l \rightarrow P_k) = \begin{cases} 0 & \text{if } r \leq l \text{ or } r > k \\ P_r & \text{if } l < r \leq k \end{cases}$$

Let $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure in $\mathcal{D}^b(A_n)$. By lemma 2.6, $\mathcal{D}^{\leq 0}$ contains a shift of the standard aisle. Without loss of generality, suppose that $\mathcal{D}_0^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}_0^{\leq 0} \not\subset \mathcal{D}^{\leq 0}$. Now let \mathcal{S} be the finite set of indecomposable objects lying in $\mathcal{D}^{\leq 0} \cap \text{mod}(A_n)$ where $\text{mod}(A_n)$ is identified with the heart of the standard t-structure. It is clear that for a negative integer k , the intersection $\mathcal{D}^{\leq 0} \cap (\text{mod}(A_n)[k])$ is contained in the k -th shift $\mathcal{S}[k]$ of \mathcal{S} . If $\mathcal{S} = \emptyset$ then the

t-structure is a shift of the standard t-structure and it is induced with respect to any recollement \mathcal{R}_r . Suppose $\mathcal{S} \neq \emptyset$. Define

$$\begin{aligned} m_0 &:= \max \{k : P_l \rightarrow P_k \in \mathcal{S}\} \\ m_1 &:= \min \{l : P_l \rightarrow P_k \in \mathcal{S}\}. \end{aligned}$$

Note that $1 \leq m_0 \leq n$ and $0 \leq m_1 \leq n-1$. We consider the following three cases:

- (1) $(m_0, m_1) \neq (n, 0)$;
- (2) $(m_0, m_1) = (n, 0)$ and $P_n \in \mathcal{S}$;
- (3) $(m_0, m_1) = (n, 0)$ and $P_n \notin \mathcal{S}$;

and we prove, case by case, the existence of a vertex r such that the given t-structure is compatible with \mathcal{R}_r , i.e., the aisle $\mathcal{D}^{\leq 0}$ is closed under the functor $j_!^r j_r^*$.

Case (1): Suppose $m_0 \neq n$ and we choose $r > m_0$ (e.g. $r = n$). For $P_l \rightarrow P_k \in \mathcal{S}$ it holds $k \leq m_0 < r$. Hence the image of \mathcal{S} by the functor $j_!^r j_r^*$ is zero. Since $\mathcal{D}_0^{\leq -1}$ is always closed under the functor $j_!^r j_r^*$, the compatibility condition holds. Suppose $m_1 \neq 0$ then we choose $r \leq m_1$ (e.g. $r = m_1$). It follows also that $\mathcal{S} \subset \text{Ker}(j_!^r j_r^*)$, and hence the compatibility condition holds.

Case (2): Suppose now $(m_0, m_1) = (n, 0)$ and $P_n \in \mathcal{S}$. Therefore, $P_s \rightarrow P_n \in \mathcal{S}$ for all $0 \leq s \leq n-1$. If, for any negative integer $k \in \mathbb{Z}_{<0}$, whenever we have $(P_s \rightarrow P_n)[k]$ for some $1 \leq s \leq n-1$ in the aisle we also have $P_n[k]$ in the aisle, then the idempotent $r = n$ fulfills the compatibility condition. Assume now that for some negative integer k , $P_n[k]$ does not belong to the aisle $\mathcal{D}^{\leq 0}$ but both $P_n[k+1]$ and $(P_s \rightarrow P_n)[k]$ (for some $1 \leq s \leq n-1$) do. We take the minimal s_0 such that $(P_{s_0} \rightarrow P_n)[k]$ belongs to $\mathcal{D}^{\leq 0}$. From the triangle

$$(P_{s_0} \rightarrow P_n)[k] \rightarrow P_{s_0}[k+1] \rightarrow P_n[k+1] \rightarrow (P_{s_0} \rightarrow P_n)[k+1]$$

we see that $P_{s_0}[k+1]$ belongs to $\mathcal{D}^{\leq 0}$. For $0 \leq t < s_0$ and $s_0 \leq m \leq n$, the triangle $(P_t \rightarrow P_m)[k] \rightarrow (P_t \rightarrow P_n)[k] \oplus (P_{s_0} \rightarrow P_m)[k] \rightarrow (P_{s_0} \rightarrow P_n)[k] \rightarrow (P_t \rightarrow P_m)[k+1]$, where $P_{s_0} \rightarrow P_m$ is viewed as zero when $s_0 = m$, shows that $(P_t \rightarrow P_m)[k]$ does not belong to $\mathcal{D}^{\leq 0}$ (because of the minimality of s_0). Note that $P_t \rightarrow P_m$ for $1 \leq t < s_0 \leq m$ are all the indecomposable modules which not killed by $j_!^{s_0} j_{s_0}^*$ and are sent to P_{s_0} . Therefore the idempotent $r = s_0$ is as desired.

Case (3): Suppose $(m_0, m_1) = (n, 0)$ and $P_n \notin \mathcal{S}$. Let

$$p_0 := \max \{k : P_k \in \mathcal{S}\} \quad \text{and} \quad p_1 := \min \{l : P_l \rightarrow P_n \in \mathcal{S}\}$$

and note that for all $l \geq p_1$, $P_l \rightarrow P_k \in \mathcal{S}$. This implies that $p_0 < p_1$ as otherwise the triangle

$$P_{p_0} \longrightarrow P_n \longrightarrow (P_{p_0} \rightarrow P_n) \longrightarrow P_{p_0}[1]$$

would imply $P_n \in \mathcal{S}$. Choose $r = p_0 + 1$. If we show that $\mathcal{S} \subset \text{Im}(i_*^r)$, then clearly $\mathcal{S} \subset \text{Ker}(j_r^*)$ and thus we get the compatibility condition as for case (1). Now, $P_l \rightarrow P_k \in \text{Im}(i_*^r)$ if and only if $l \geq r$ or $k < r$. Suppose $P_l \rightarrow P_k \in \mathcal{S}$ such that $0 < l < r$ and $k \geq r$. As above, since $1 \leq l \leq p_0 \leq k-1$, there is a triangle

$$P_{p_0} \longrightarrow P_k \oplus (P_l \rightarrow P_{p_0}) \longrightarrow (P_l \rightarrow P_k) \longrightarrow P_{p_0}[1]$$

implying that $P_k \in \mathcal{S}$ - a contradiction with the choice of p_0 . Thus $\mathcal{S} \subset \text{Im}(i_*^r)$. \square

The factors appearing in such recollements \mathcal{R}_r for $\mathcal{D}^b(A_n)$ are either $\mathcal{D}^b(\mathbb{K})$, $\mathcal{D}^b(A_k)$ for some $k < n$ or $\mathcal{D}^b(A_k \times A_p)$ for some k and p with $k+p < n$. This theorem shows that we can inductively construct all bounded t-structures in $\mathcal{D}^b(A_n)$. Furthermore we only need the recollements coming from these idempotents.

As a corollary one can list all bounded t-structures for $\mathcal{D}^b(A_2)$. Given the simplicity of computations for A_2 we can, indeed, go one step further and describe all t-structures, without the boundedness constraint. Recall that for a triangulated category \mathcal{D} endowed with a recollement \mathcal{R} , we write $\mathcal{T}_{\mathcal{D}}^{\mathcal{R}}$ for the set of t-structures in \mathcal{D} which are compatible with the recollement \mathcal{R} .

Proposition 5.2. *If $\mathcal{D}^{\leq 0} \subset \mathcal{D}^b(A_2)$ is an aisle of a t-structure in $\mathcal{T}_{\mathcal{D}^b(A_2)}^{\mathcal{R}_1} \cup \mathcal{T}_{\mathcal{D}^b(A_2)}^{\mathcal{R}_2}$, then it can be described by one of the following types:*

- (1) *It is trivial (i.e., equal to 0 or to $\mathcal{D}^b(A_2)$);*
- (2) *It is a shift of the standard aisle;*
- (3) *Its indecomposable objects are (some or all) shifts of P_1 ;*
- (4) *Its indecomposable objects are (some or all) shifts of S_2 ;*
- (5) *Its indecomposable objects are (some or all) shifts of P_2 ;*
- (6) *Its indecomposable objects are the union of the indecomposable objects of a shift of a standard aisle with (some or all) shifts of P_1 ;*
- (7) *Its indecomposable objects are the union of the indecomposable objects of a shift of a standard aisle with (some or all) shifts of S_2 ;*
- (8) *Its indecomposable objects are the union of the indecomposable objects of a shift of a standard aisle with (some or all) shifts of P_2 and exactly one extra shift of S_2 .*

Furthermore these are all the aisles of $\mathcal{D}^b(A_2)$.

Proof. Let A be the path algebra of the quiver A_2 . Note that, as rings, $A/Ae_rA \cong \mathbb{K} \cong e_rAe_r$ for $r = 1, 2$. By Lemma 4.2 $\mathcal{T}_{\mathcal{D}^b(\mathbb{K})} \cong \bar{\mathbb{Z}}$. We check first that these t-structures can occur as BBD-induction from pairs. We will use the notation $\mathcal{D}_{(r;m,n)}^{\leq 0}$ to describe the aisle resulting from induction of the pair $(m, n) \in \bar{\mathbb{Z}} \times \bar{\mathbb{Z}} \cong \mathcal{T}_{\mathcal{D}^b(A/Ae_rA)} \times \mathcal{T}_{\mathcal{D}^b(e_rAe_r)}$ with respect to the recollement \mathcal{R}_r ($r = 1, 2$). The following statements are easy to check using the explicit functors in \mathcal{R}_r :

- (1) If $\mathcal{D}^{\leq 0}$ is trivial (i.e., equal to 0 or to $\mathcal{D}^b(A_2)$) then we have that $\mathcal{D}^{\leq 0}$ is equal to either $\mathcal{D}_{(r;-\infty,-\infty)}^{\leq 0}$ or $\mathcal{D}_{(r;+\infty,+\infty)}^{\leq 0}$;
- (2) If $\mathcal{D}^{\leq 0}$ is a shift of the standard aisle, it is equal to $\mathcal{D}_{(r;n,n)}^{\leq 0}$;
- (3) If the indecomposable objects of $\mathcal{D}^{\leq 0}$ are shifts of P_1 then we have that $\mathcal{D}^{\leq 0}$ is equal to either $\mathcal{D}_{(1;+\infty,-\infty)}^{\leq 0} = \mathcal{D}_{(2;-\infty,+\infty)}^{\leq 0}$ (if all shifts are there) or $\mathcal{D}_{(1;n,-\infty)}^{\leq 0} = \mathcal{D}_{(2;-\infty,n)}^{\leq 0}$ (if only some shifts are there);
- (4) If the indecomposable objects of $\mathcal{D}^{\leq 0}$ are shifts of S_2 then we have that $\mathcal{D}^{\leq 0}$ is equal to either $\mathcal{D}_{(1;-\infty,+\infty)}^{\leq 0}$ (if all shifts are there) or $\mathcal{D}_{(1;-\infty,n)}^{\leq 0}$ (if only some shifts are there);
- (5) If the indecomposable objects of $\mathcal{D}^{\leq 0}$ are shifts of P_2 then we have that $\mathcal{D}^{\leq 0}$ is equal to either $\mathcal{D}_{(2;+\infty,-\infty)}^{\leq 0}$ (if all shifts are there) or $\mathcal{D}_{(2;n,-\infty)}^{\leq 0}$ (if only some shifts are there);
- (6) If the indecomposable objects of $\mathcal{D}^{\leq 0}$ are the union of the indecomposable objects of a shift of a standard aisle with shifts of P_1 then we have that $\mathcal{D}^{\leq 0}$ is equal to either $\mathcal{D}_{(1;+\infty,n)}^{\leq 0} = \mathcal{D}_{(2;n,+\infty)}^{\leq 0}$ (if all shifts are there) or $\mathcal{D}_{(1;n,m)}^{\leq 0} = \mathcal{D}_{(2;m,n)}^{\leq 0}$ for $m < n$ (if only some shifts are there);
- (7) If the indecomposable objects of $\mathcal{D}^{\leq 0}$ are the union of the indecomposable objects of a shift of a standard aisle with shifts of S_2 then we have that $\mathcal{D}^{\leq 0}$

- is equal to either $\mathcal{D}_{(1;m,+\infty)}^{\leq 0}$ (if all shifts are there) or $\mathcal{D}_{(1;m,n)}^{\leq 0}$ for $m < n$ (if only some shifts are there);
- (8) If the indecomposable objects of $\mathcal{D}^{\leq 0}$ are the union of the indecomposable objects of a shift of a standard aisle with shifts of P_2 and exactly one extra shift of S_2 then we have that $\mathcal{D}^{\leq 0}$ is equal to either $\mathcal{D}_{(2;+\infty,n)}^{\leq 0}$ (if all shifts are there) or $\mathcal{D}_{(2;m,n)}^{\leq 0}$ for $m > n$ (if only some shifts are there).

We now prove that these are all possible aisles. Suppose $\mathcal{D}^{\leq 0}$ is neither zero nor the whole category (i.e. not of type (1)). Then it contains some indecomposable objects. If the aisle contains shifts of only one indecomposable module, then all possibilities are listed in the proposition - they are types (3),(4) and (5). So we suppose there are at least two indecomposable objects which are shifts of distinct indecomposable modules. This implies that a standard aisle is contained in $\mathcal{D}^{\leq 0}$, since we will have at least one triangle of indecomposable objects in $\mathcal{D}^{\leq 0}$ which is then necessarily a shift of

$$P_1 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow P_1[1].$$

We assume, without loss of generality, that $\mathcal{D}_0^{\leq -1} \subset \mathcal{D}^{\leq 0}$ but $\mathcal{D}_0^{\leq 0} \not\subset \mathcal{D}^{\leq 0}$. If $\mathcal{D}_0^{\leq -1} = \mathcal{D}^{\leq 0}$ we are done (type (2)), otherwise, by considering the triangle above we observe the following:

- if $S_2 \in \mathcal{D}^{\leq 0}$ then $P_1 \notin \mathcal{D}^{\leq 0}$ (otherwise $S_2 \in \mathcal{D}^{\leq 0}$ and thus $\mathcal{D}_0^{\leq 0} = \mathcal{D}^{\leq 0}$). If $P_2 \notin \mathcal{D}^{\leq 0}$ then the only remaining indecomposable objects that can lie in the aisle are the negative shifts of S_2 and this is type (7). If $P_2 \in \mathcal{D}^{\leq 0}$ then $S_2[-1] \notin \mathcal{D}^{\leq 0}$ (otherwise $P_1 \in \mathcal{D}^{\leq 0}$ and thus $\mathcal{D}_0^{\leq 0} = \mathcal{D}^{\leq 0}$) and thus the only remaining indecomposable objects that can lie in the aisle are negative shifts of P_2 and this is type (8);
- if $P_2 \in \mathcal{D}^{\leq 0}$, then so is S_2 and we fall on the previous case;
- If $P_1 \in \mathcal{D}^{\leq 0}$, then $P_2, S_2 \notin \mathcal{D}^{\leq 0}$ and the only remaining indecomposable objects that can lie in the aisle are the negative shifts of P_1 - and this is type (6).

This concludes the proof. \square

We end this section by two clarifying remarks making use of the explicit simple nature of $\mathcal{D}^b(A_2)$.

Remark 5.3. The proof above shows how different recollements allow different types of induced t-structures. While \mathcal{R}_1 and \mathcal{R}_2 both allow types (1), (2), (3), and (6) to appear as BBD-induction, only \mathcal{R}_1 induces types (4) and (7) and only \mathcal{R}_2 induces types (5) and (8).

Remark 5.4. Also it is not hard to check the equivalence classes for bounded t-structures in $\mathcal{D}^b(A_2)$. Indeed, by the work of Miyachi and Yekutieli ([18]), the group of autoequivalences of A_2 is known to be generated by the Auslander-Reiten translation τ and the triangulated shift [1]. By explicitly computing the orbits of a bounded t-structure in the list above, one can easily see that the equivalence class of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is determined by the number of connected components of the intersection of the Auslander-Reiten quiver of $\mathcal{D}^b(A_2)$ with $\mathcal{D}^{\leq 0}$. Therefore, the set of equivalence classes of bounded t-structures for $\mathcal{D}^b(A_2)$ is naturally parametrized by \mathbb{N} .

6. BOUNDED T-STRUCTURES FOR PIECEWISE HEREDITARY ALGEBRAS

In this section we generalize our result of the previous section to piecewise hereditary algebras, and prove that any bounded t-structure with length heart is a BBD-induction with respect to some recollement by derived categories. Moreover when the algebra is hereditary of finite representation type, we prove that any bounded t-structure has a length heart, and hence it is always a BBD-induction. Indeed, since we are only interested in recollements by derived categories (and not general triangulated categories, see [2] for a discussion), by the expression *a t-structure is BBD-induced* we will mean *a t-structure is BBD-induced with respect to some recollement by derived categories*.

Let A be a finite dimensional \mathbb{K} -algebra. Recall that A is *piecewise hereditary*, if there exists a hereditary and abelian \mathbb{K} -category \mathcal{H} such that the bounded derived categories $\mathcal{D}^b(A)$ and $\mathcal{D}^b(\mathcal{H})$ are triangle equivalent. In other words, there exists a tilting complex T in $\mathcal{D}^b(\mathcal{H})$ with endomorphism ring being A . In particular, the derived category $\mathcal{D}^b(A)$ is hereditary, i.e., each indecomposable object is a stalk complex concentrated in one component, or equivalently a shift of an indecomposable object in \mathcal{H} .

Given a family of simple-minded objects X_1, \dots, X_n in $\mathcal{D}^b(A)$, by Subsection 2.4, their extension closure is a module category of a certain finite dimensional algebra Γ . In the piecewise hereditary case, this algebra Γ is even directed, in the sense that the quiver of Γ has no oriented cycles, or equivalently, there are no cycles

$$P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_s = P_1$$

of nonzero homomorphisms (which are not isomorphisms) between projective indecomposable Γ -modules. This follows from Happel [9, Lemma IV.1.10]. For the convenience of the reader we include a proof here.

Lemma 6.1 ([9] Lemma IV.1.10). *Let A be a piecewise hereditary algebra over \mathbb{K} , and X_1, \dots, X_n a family of simple-minded objects in $\mathcal{D}^b(A)$. Then the extension closure of X_i 's in $\mathcal{D}^b(A)$ is equivalent to $\text{mod}(\Gamma)$ for some finite dimensional directed algebra Γ . In particular, X_i is exceptional (i.e. has no self-extensions).*

Proof. Let \mathcal{H} be an abelian and hereditary category \mathcal{H} with $\mathcal{D}^b(A) \cong \mathcal{D}^b(\mathcal{H})$. By Happel and Reiten [10] and Lenzing [16], \mathcal{H} can be chosen to be either the module category $\text{mod}(H)$ of some finite dimensional hereditary algebra H , or $\text{coh}(X)$ the category of coherent sheaves over some exceptional curve X . In these two cases, given indecomposable objects $X, Y \in \mathcal{H}$ with $\text{Ext}_{\mathcal{H}}^1(Y, X) = 0$, then a nonzero homomorphism from X to Y is either a monomorphism or an epimorphism ([13, 4.1]). In particular if X has no self-extensions then the endomorphism ring $\text{End}_{\mathcal{H}}(X)$ is a division ring.

Assume now $P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_s = P_1$ is a cycle of indecomposable projective Γ -modules. View this as a cycle in the derived category $\mathcal{D}^b(A) \cong \mathcal{D}^b(\mathcal{H})$. Because \mathcal{H} is hereditary, there exists an integer k such that all P_i 's belong to the k -th shift $\mathcal{H}[k]$ ([9, I.5.3]). Without loss of generality assume $k = 0$. For any $i, j = 1, \dots, s$, by Lemma 2.2, $0 = \text{Ext}_{\Gamma}^1(P_i, P_j) = \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(P_i, P_j[1])$, which is isomorphic to $\text{Ext}_{\mathcal{H}}^1(P_i, P_j)$. It follows that every map occurring in the cycle is either a monomorphism or epimorphism. Indeed, either all maps are monomorphisms or all maps are epimorphisms (otherwise we would get a nonzero map which is a composition

of a proper monomorphism with a proper epimorphism but neither a monomorphism nor an epimorphism). Since $\text{End}(P_1)$ is a division ring, all maps must be isomorphisms, contradiction to the definition of a cycle.

Finally because of the directedness of Γ each X_i has no self-extensions as a Γ -module. In particular by Lemma 2.2, $\text{Hom}_{\mathcal{D}^b(A)}(X_i, X_i[1]) = \text{Ext}_{\Gamma}^1(X_i, X_i) = 0$. Since $\mathcal{D}^b(A) \cong \mathcal{D}^b(\mathcal{H})$ is hereditary, no higher self-extensions are possible. \square

Before we present our main result, we need one further result, showing how to construct a recollement of a piecewise hereditary algebra from an indecomposable and *exceptional* object (i.e., an object without self-extensions).

Proposition 6.2 ([2] Theorem 2.5; [3] Proposition 5.8). *Let A be a piecewise hereditary algebra over a field \mathbb{K} . For any indecomposable and exceptional object X in $\mathcal{D}^b(A)$, there exists a recollement of the form*

$$\mathcal{D}^b(B) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^b(A) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^b(C) .$$

where B and C are again piecewise hereditary algebras with $C = \text{End}_A(X)$ being a division ring over \mathbb{K} .

By [10], a quasi-hereditary algebra is derived equivalent to either a hereditary algebra or a canonical algebra. These two cases are discussed by [2, Theorem 2.5] and [3, Proposition 5.8] respectively. We can now prove our main result.

Theorem 6.3. *Let A be a piecewise hereditary algebra over a field \mathbb{K} and suppose $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a bounded t -structure in $\mathcal{D}^b(A)$ whose heart is a length category. Then T is compatible with a recollement of the form*

$$\mathcal{D}^b(B) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^b(A) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathcal{D}^b(C) .$$

(and hence a BBD-induction with respect to it) for some piecewise hereditary algebras B and C . Moreover, C can always be taken to be a division ring over the base field \mathbb{K} .

Proof. Write $\mathcal{D} = \mathcal{D}^b(A)$ for short. Let T be a bounded t -structure in \mathcal{D} with heart \mathcal{A} being a length category. Take a set of simple objects X_1, \dots, X_n in \mathcal{A} . By Theorem 2.5, there is a finite dimensional algebra Γ such that $\mathcal{A} \cong \text{mod}(\Gamma)$ and X_1, \dots, X_n are simple Γ -modules. Since A is piecewise hereditary, Lemma 6.1 shows that, moreover, Γ is directed. Let \mathcal{H} be an abelian and hereditary category with $\mathcal{D}^b(\mathcal{H}) \cong \mathcal{D}$. There exists integer $l_i \in \mathbb{Z}$ such that $X_i \in \mathcal{H}[l_i]$ the l_i -th shift (for $i = 1, \dots, n$). Because of the directness of Γ , we can assume without loss of generality that X_1 is the simple projective Γ -module with l_1 smallest among simple projective Γ -modules. We claim that $l_i \geq l_1$ for all $2 \leq i \leq n$. When X_i is again simple projective as Γ -module, because of the choice of l_1 it holds $l_i \geq l_1$. When X_i is not projective, there must exists some X_j , simple projective as Γ -module, and a path in $\text{mod}(\Gamma)$ from X_j to X_i . View this path in the derived category $\mathcal{D}^b(\mathcal{H})$ and we see that $l_i \geq l_j$. Also $l_j \geq l_1$ because of the choice of l_1 . Hence $l_i \geq l_1$ for all $2 \leq i \leq n$.

Next we show that X_2, \dots, X_n belong to the right perpendicular category of X_1 , that is, $\text{Hom}_{\mathcal{D}}(X_1, X_i[k]) = 0$ for $2 \leq i \leq n$ and for all integers $k \in \mathbb{Z}$. According to the definition of simple-minded objects, the Hom-set vanishes whenever $k \leq 0$. We

only have to concern with the case $k > 0$. Let M_i be the indecomposable objects in \mathcal{H} with $X_i = M_i[l_i]$. We have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(X_1, X_i[k]) &= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{H})}(M_1[l_1], M_i[l_i + k]) \\ &= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{H})}(M_1, M_i[l_i - l_1 + k]) \\ &= \mathrm{Ext}_{\mathcal{H}}^{l_i - l_1 + k}(M_1, M_i), \end{aligned}$$

which is zero whenever $l_i - l_1 + k \neq 0, 1$. Under the condition that $l_i \geq l_1$ and $k > 0$, the only possible nontrivial case is $l_i = l_1$ and $k = 1$. In this case $\mathrm{Hom}_{\mathcal{D}}(X_1, X_i[1]) = \mathrm{Ext}_{\Gamma}^1(X_1, X_i)$, by Lemma 2.2, which is zero because X_1 is a projective Γ -module.

Note that X_1 is indecomposable and exceptional in $\mathcal{D}^b(A)$. By Proposition 6.2, there exists a recollement of $\mathcal{D}^b(A)$ generated by X_1

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}^b(B) & \xrightarrow{i_*} & \mathcal{D}^b(A) & \xrightarrow{j^*} & \mathcal{D}^b(C) \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array}.$$

where B and C are again piecewise hereditary algebras. Indeed $C = \mathrm{End}_{\mathcal{D}^b(A)}(X_1)$ is a division ring, $\mathrm{Im}(j_!) = \mathrm{tria}(X_1) = \{X_1[k] : k \in \mathbb{Z}\}$ and $\mathrm{Im}(i_*) = X_1^\perp := \{Y \in \mathcal{D}^b(A) : \mathrm{Hom}_{\mathcal{D}^b(A)}(X_1, Y[k]) = 0, \forall k \in \mathbb{Z}\}$. We will show that the t-structure T is compatible with this recollement, i.e., that $j_!j^*(\mathcal{D}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$. Corollary 3.4 shows that T is then a BBD-induction.

Since X_1, \dots, X_n is the set of simple-minded objects corresponding to T , the aisle $\mathcal{D}^{\leq 0}$ is just the extension closure of non-negative shifts of X_i 's. We have showed that X_2, \dots, X_n are right perpendicular to X_1 , and hence belong to $\mathrm{Im}(i_*)$. They are sent to zero by j^* . It follows that the indecomposable objects of $j_!j^*(\mathcal{D}^{\leq 0})$ are exactly $X_1[k]$ for $k \geq 0$, which form a subset of the aisle $\mathcal{D}^{\leq 0}$. This finishes our proof. \square

Combined with Proposition 3.9 we obtain the following immediate following.

Corollary 6.4. *Let A be a piecewise hereditary algebra over a field \mathbb{K} , and $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ a bounded t-structure in $\mathcal{D}^b(A)$. Then the heart of T is a length category if and only if T is BBD-induced from t-structures with length hearts.*

Remark 6.5. In [3] a Jordan Hölder theorem for derived categories of piecewise hereditary algebras is proven. If A is piecewise hereditary over a field \mathbb{K} , the derived-simple factors of $\mathcal{D}^b(A)$ are of the form $\mathcal{D}^b(\mathbb{D})$ where \mathbb{D} is a division ring over \mathbb{K} . By Lemma 4.2 every bounded t-structure of $\mathcal{D}^b(\mathbb{D})$ has a length heart. Hence the bounded t-structures of $\mathcal{D}^b(A)$ with length hearts are precisely the ones obtained by BBD-induction.

Example 6.6. Let A be the path algebra of the Kroenecker quiver \mathbb{C} . Then it is well known that $\mathcal{D}^b(A) \cong \mathcal{D}^b(\mathrm{coh}(\mathbb{P}^1))$ and therefore $\mathrm{coh}(\mathbb{P}^1)$ is the heart of a bounded t-structure in $\mathcal{D}^b(A)$. It is not, however, a length heart. Then, since A is hereditary, Corollary 6.4 shows that this bounded t-structure cannot be induced with respect to any recollement by derived categories.

For the rest of the section, we consider hereditary algebras of finite representation type.

Lemma 6.7. *Let A be a hereditary algebra of finite representation type, and T any bounded t-structure in $\mathcal{D}^b(A)$. Then the heart of T is a length category.*

Proof. Suppose T is a bounded t-structure in $\mathcal{D}^b(A)$ with heart \mathcal{A} . If X is an indecomposable object in \mathcal{A} , then any nonzero shift $X[k]$ ($k \neq 0$) can not lie in \mathcal{A} since $\mathcal{A}[k] \cap \mathcal{A} = 0$ by definition of heart. But indecomposable objects in $\mathcal{D}^b(A)$ are stalk complexes and A has finite representation type. Hence \mathcal{A} contains only finitely many indecomposable objects. Assume there exists some object X in \mathcal{A} with infinite length. That means it admits an infinite sequence of monomorphisms $\dots X_{-1} \rightarrow X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X$. Write each X_i as a direct sum of indecomposable objects. We will be able to find a cycle of indecomposable objects. But on the other hand, since A is hereditary of finite representation type, we know there exists a total order on indecomposable objects in $\text{mod}(A)$, and hence in $\mathcal{D}^b(A)$, such that $\text{Hom}(X_j, X_i) = 0$ whenever $j > i$. A contradiction! Therefore the heart \mathcal{A} must be a length category. \square

Combined with Theorem 2.5, we obtain a bijection between the set of bounded t-structures in $\mathcal{D}^b(A)$ and equivalence classes of families of simple-minded objects. It follows from Theorem 6.3 that any bounded t-structure can be induced from some recollement. In fact the next result shows that up to Auslander-Reiten translation, which is an autoequivalence of $\mathcal{D}^b(A)$, we can choose the recollement to be of type \mathcal{R}_r (i.e. associated with an idempotent e_r of A , see the beginning of Section 5).

Corollary 6.8. *Let A be a hereditary algebra of finite representation type, and $T = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ any bounded t-structure in $\mathcal{D}^b(A)$. Then there exists an autoequivalence Φ of $\mathcal{D}^b(A)$ such that $\Phi(T) := (\Phi(\mathcal{D}^{\leq 0}), \Phi(\mathcal{D}^{\geq 0}))$ is compatible with (and hence a BBD-induction with respect to) a recollement of type \mathcal{R}_r for some r .*

Proof. Let X_1, \dots, X_n be the family of simple-minded objects associated to the bounded t-structure T . First of all we give an alternative way of finding such an X_1 as in the proof of Theorem 6.3. For the algebra A is hereditary of finite representation type, as mentioned in the proof of Lemma 6.7, there exists a total order on the set of indecomposable objects in $\mathcal{D}^b(A)$ such that $X < X[1]$ for any indecomposable X , and $\text{Hom}(X, Y) \neq 0$ implies $X < Y$ for any indecomposable X and Y . Without loss of generality we assume that $X_1 < X_i$ for all $i \geq 1$. We have to show that X_i , for all $i \geq 2$, belongs to the right perpendicular category $X_1^\perp = \{Y \in \mathcal{D}^b(A) : \text{Hom}(X, Y[k]) = 0, \forall k \in \mathbb{Z}\}$ of X_1 . Since X_1, \dots, X_n is a family of simple-minded objects, $\text{Hom}(X_1, X_i[k]) = 0$ for all $i \geq 2$ and all $k \leq 0$. Assume $\text{Hom}(X_1, X_i[k]) \neq 0$ for some $i \geq 2$ and some $k > 0$. Then we have a triangle $X_i[k-1] \rightarrow Y \rightarrow X_1 \rightarrow X_i[k]$ with $k-1 \geq 0$. It follows that $X_i \leq X_i[k-1] < Y < X_1$, a contradiction with the choice of X_1 .

Again because A is hereditary of finite representation type, the Auslander-Reiten translation τ is an autoequivalence of $\mathcal{D}^b(A)$. Moreover, any indecomposable object in $\mathcal{D}^b(A)$ can be transformed to an indecomposable projective A -module by iteratively applying τ . So there exists a natural number $s \in \mathbb{N}$ and an idempotent $e_r \in A$ such that $\tau^s(X_1) = P_r$, the indecomposable projective A -module associated to e_r . It is clear that $\tau^s(T) = (\tau^s(\mathcal{D}^{\leq 0}), \tau^s(\mathcal{D}^{\geq 0}))$ is still a bounded t-structure in $\mathcal{D}^b(A)$, $(\tau^s(X_1), \dots, \tau^s(X_n))$ is the associated family of simple-minded objects, and $\tau^s(X_i)$, for $i \geq 2$, belong to the right perpendicular category P_r^\perp . By Theorem 6.3, $\Phi(T)$ is a BBD-induction with respect to the recollement of $\mathcal{D}^b(A)$ generated by P_r . This recollement is precisely of type \mathcal{R}_r . \square

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